Nontrivial independent sets of bipartite graphs and cross-intersecting families

Jun Wang [a] and Huajun Zhang [a,b,1]

[a] *Department of Mathematics, Shanghai Normal University, Shanghai 200234, P.R. China*

[b] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P.R. China*

Abstract

Let $G(X,Y)$ be a connected, non-complete bipartite graph with $|X| \leq |Y|$. An independent set A of $G(X,Y)$ is said to be trivial if $A \subseteq X$ or $A \subseteq Y$. Otherwise, A is nontrivial. By $\alpha(X,Y)$ we denote the size of maximal-sized nontrivial independent sets of $G(X,Y)$. We prove that if the automorphism group of $G(X,Y)$ is transitive on X and Y, then $\alpha(X,Y) = |Y| - d(X) + 1$, where $d(X)$ is the common degree of vertices in X. We also give the structures of maximal-sized nontrivial independent sets of $G(X,Y)$. As applications of this result, we give the upper bound of sizes of two cross-t-intersecting families of finite sets, finite vector spaces and permutations.

Key words: intersecting family, cross-intersecting family, symmetric system, Erdős-Ko-Rado theorem

MSC: 05D05, 06A07

1 Introduction

Let X be a finite set and, for $0 \leq k \leq |X|$, let $\binom{X}{k}$ denote the family of all k-subsets of X, and let S_X and A_X denote the symmetric group and alternative group on X, respectively. In particular, for positive integer n, let

Email addresses: jwang@shnu.edu.cn (Jun Wang), huajunzhang@zjnu.cn (Huajun Zhang).

[1] Corresponding author.

$[n]$ denote the set $\{1, 2, \ldots, n\}$, $[k, n] = \{k+1, \ldots, n\}$ for $k \le n$, $\overline{A} = [n] \backslash A$ for $A \subseteq [n]$, and abbreviate the symmetric group and alternative group on $[n]$ as S_n and A_n, respectively.

A family $\mathcal{A}$ of sets is said to be t-intersecting if $|A \cap B| \ge t$ holds for all $A, B \in \mathcal{A}$. Usually, $\mathcal{A}$ is called intersecting if $t = 1$. The celebrated Erdős–Ko–Rado theorem [11], says that if $\mathcal{A}$ is a t-intersecting family in $\binom{[n]}{k}$, then

$$|\mathcal{A}| \le \binom{n-t}{k-t}$$

for $n \ge n_0(k, t)$. The smallest $n_0(k, t) = (k - t + 1)(t + 1)$ was determined by Frankl [12] for $t \ge 15$ and subsequently determined by Wilson [29] for all t.

The Erdős-Ko-Rado theorem has many generalizations, analogs and variations. First, the notion of intersection is generalized to t-intersection, and finite sets are analogous to finite vector spaces, permutations and other mathematical objects. Second, intersecting families are generalized to cross-intersecting families: $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are said to be cross-t-intersecting if $|A \cap B| \ge t$ for all $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, $i \ne j$. Some typical but far from exhaustive results are listed as follows.

Let $\mathbb{F}_q$ be a finite field of order q, $V = V_n(q)$ an n-dimensional vector space over $\mathbb{F}_q$, and $\begin{bmatrix} V \\ k \end{bmatrix}$ the set of all k-dimensional subspaces (or k-subspace, for short) of V. Then the cardinality of $\begin{bmatrix} V \\ k \end{bmatrix}$ equals $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}$. For brevity, we write $\begin{bmatrix} n \\ k \end{bmatrix}$ rather than $\begin{bmatrix} n \\ k \end{bmatrix}_q$. A subset $\mathcal{A}$ of $\begin{bmatrix} V \\ k \end{bmatrix}$ is said to be a t-intersecting family if $\dim(A \cap B) \ge t$ for any $A, B \in \mathcal{A}$. The Erdős-Ko-Rado theorem for finite vector spaces says that if $\mathcal{A}$ is a t-intersecting family in $\begin{bmatrix} V \\ k \end{bmatrix}$, then

$$|\mathcal{A}| \le \max\left\{ \begin{bmatrix} n-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix} \right\}$$

for $n \ge 2k - t$. This theorem was first established by Hsieh [19] for $t = 1$ and $k < n/2$, then by Greene and Kleitman [15] for $t = 1$ and $k|n$, and finally by Frankl and Wilson [14] for the general case.

A subset A of S_n is said to be a t-intersecting family if any two permutations in A agree in at least t points, i.e. for any $\sigma, \tau \in A$, $|\{i \in [n] : \sigma(i) = \tau(i)\}| \ge t$. Deza and Frankl [10] showed that an intersecting family in S_n has size at most $(n-1)!$ and conjectured that for t fixed, and n sufficiently large depending on t, a t-intersecting family in S_n has size at most $(n-t)!$. Cameron and Ku

[8] proved an intersecting family of size $(n-1)!$ is a coset of the stabilizer of a point. A few alternative proofs of Cameron and Ku's result are given in [23], [16] and [27]. Ku and Leader [22] also generalized this result to partial permutations (see also [24]). Ellis, Friedgut and Pilpel [2] proved Deza and Frankl's conjecture on t-intersecting family in S_n.

Hilton [17] investigated the cross-intersecting families in $\binom{[n]}{k}$: Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be cross-intersecting families in $\binom{[n]}{k}$ with $\mathcal{A}_1 \neq \emptyset$. If $k \leq n/2$, then

$$\sum_{i=1}^{m} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{k}, & \text{if } m \leq \frac{n}{k}; \\ m\binom{n-1}{k-1}, & \text{if } m \geq \frac{n}{k}. \end{cases} \tag{1}$$

He also determined the structures of $\mathcal{A}_i$'s when equality holds. Borg [5] gives a simple proof of this theorem, and generalizes it to labeled sets [4], signed sets [7] and permutations [6]. We generalized this theorem to general symmetric systems [26], which contain finite sets, finite vector spaces and permutations, etc.

Hilton and Milner [18] and Frankl and Tokushige [13] also investigated the sizes of two cross-intersecting families: If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting families with $n \geq a+b$, $a \leq b$, then $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{b} - \binom{n-a}{b} + 1$.

This theorem actually gives a upper bound of the sizes of nontrivial independent sets in a bipartite graph.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, define $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G(A) = \cup_{v \in A} N_G(v)$ for $A \subseteq V(G)$. If there is no possibility of confusion, we abbreviate $N_G(A)$ as $N(A)$. A subset A of $V(G)$ is an independent set of G if $A \cap N(A) = \emptyset$. A graph G is bipartite if $V(G)$ can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. In this case, we denote the bipartite graph by $G(X,Y)$. An independent set A of $G(X,Y)$ is said to be trivial if $A \subseteq X$ or $A \subseteq Y$. In any other case, A is nontrivial. If every vertex in X is adjacent to every vertex in Y, then $G(X,Y)$ is called a complete bipartite graph. Clearly, a complete bipartite graph has only trivial independent sets. A bipartite graph $G(X,Y)$ is said to be *part-transitive* if there is a group Γ transitively acting on X and Y, respectively, and preserving the adjacency relation of the graph. Clearly, if $G(X,Y)$ is part-transitive, then every vertex of X (Y) has the same degree, written as $d(X)$ $(d(Y))$. By $\alpha(X,Y)$ and $I(X,Y)$

we denote the size and the set of maximal-sized nontrivial independent sets of $G(X, Y)$, respectively.

This paper contributes to $\alpha(X, Y)$ and $I(X, Y)$ for part-transitive bipartite graphs $G(X, Y)$. To do this we make a simple observation as follows.

Let $G(X, Y)$ be a non-complete bipartite graph and let $A \cup B$ be a nontrivial independent set of $G(X, Y)$, where $A \subset X$ and $B \subset Y$. Then $A \subseteq X \backslash N(B)$ and $B \subseteq Y \backslash N(A)$, which implies that

$$|A| + |B| \leq \max\{|A| + |Y| - |N(A)|, |B| + |X| - |N(B)|\}.$$

From this one sees that

$$\alpha(X, Y) = \max\{|Y| - \epsilon(X), |X| - \epsilon(Y)\}, \tag{2}$$

where

$$\epsilon(X) = \min\{|N(A)| - |A| : A \subset X \text{ and } N(A) \neq Y\}$$

and

$$\epsilon(Y) = \min\{|N(B)| - |B| : B \subset Y \text{ and } N(B) \neq X\}.$$

A subset A of X is called a *fragment* in X if $N(A) \neq Y$ and $|N(A)| - |A| = \epsilon(X)$. By $\mathcal{F}(X)$ we denote the set of all fragments contained in X. $\mathcal{F}(Y)$ is defined in a similar way and write $\mathcal{F}(X, Y) = \mathcal{F}(X) \cup \mathcal{F}(Y)$. An element $A \in \mathcal{F}(X, Y)$ is also called a k-fragment if $|A| = k$. As we shall see (Lemma 2.1) that $|Y| - \epsilon(X) = |X| - \epsilon(Y)$. Therefore, in order to address our problems it suffices to determine $\mathcal{F}(X)$ or $\mathcal{F}(Y)$.

Let X be a finite set, and Γ a group transitively acting on X. We say the action of Γ on X is *primitive*, or Γ is primitive on X, if Γ preserves no nontrivial partition of X. In any other case, the action of Γ is *imprimitive*. It is easy to see that if the action of Γ on X is transitive and imprimitive, then there is a subset B of X such that $1 < |B| < |X|$ and $\gamma(B) \cap B = B$ or $\emptyset$ for every $\gamma \in G$. In this case, B is called an *imprimitive set* in X. It is well known that the action of Γ is primitive if and only if for each $a \in X$, the stabilizer of a, written as Γ_a defined to be the set $\{\gamma \in \Gamma : \gamma(a) = a\}$, is a maximal subgroup of Γ (cf. [20, Theorem 1.12]). Furthermore, a subset B of X is said to be *semi-imprimitive* if $1 < |B| < |X|$ and $|\gamma(B) \cap B| = 0, 1$ or $|B|$ for each $\gamma \in \Gamma$. Clearly, every 2-subset of X is semi-imprimitve.

The following are main results of this paper.

Theorem 1.1 *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| \leq |Y|$. If $G(X,Y)$ is part-transitive and every fragment in X and Y is primitive under the action of a group Γ. Then $\alpha(X,Y) = |Y| - d(X) + 1$. Moreover,*

(i) *if $|X| < |Y|$, then each fragment in X has size 1;*
(ii) *if $|X| = |Y|$, then each fragment in X has size 1 or $|X| - d(X)$ unless there is a semi-imprimitive fragment in X or Y.*

As consequences of this theorem we give the upper bounds of sizes of two cross-t-intersecting families of finite sets, finite vector spaces and symmetric groups.

Theorem 1.2 *Let n, a, b, t be positive integers with $n \geq 4$, $a, b \geq 2$, $t < \min\{a,b\}$, $a + b < n + t$, $(n,t) \neq (a+b, 1)$ and $\binom{n}{a} \leq \binom{n}{b}$. If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-t-intersecting, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{b} - \sum_{i=0}^{t-1} \binom{a}{i}\binom{n-a}{b-i} + 1. \tag{3}$$

Moreover,

(i) *when $\binom{n}{a} < \binom{n}{b}$, equality holds if and only if $\mathcal{A} = \{A\}$ and $\mathcal{B} = \binom{[n]}{b} \backslash N(A)$ for any $A \in \binom{[n]}{a}$;*
(ii) *when $\binom{n}{a} = \binom{n}{b}$, equality holds if and only if either $\mathcal{A} = \{A\}$ and $\mathcal{B} = \binom{[n]}{b} \backslash N(A)$ for any $A \in \binom{[n]}{a}$, or $\mathcal{B} = \{B\}$ and $\mathcal{A} = \binom{[n]}{a} \backslash N(B)$ for any $B \in \binom{[n]}{b}$, or $\{a, b, t\} = \{2, 2, 1\}$ and $\mathcal{A} = \mathcal{B} = \{C \in \binom{[n]}{2} : i \in C\}$ for some $i \in [n]$, or $\{a, b, t\} = \{n-2, n-2, n-3\}$ and $\mathcal{A} = \mathcal{B} = \binom{A}{n-2}$ for some $A \in \binom{[n]}{n-1}$.*

Theorem 1.3 *Let V be an n-dimensional vector space over the field of order q and let n, a, b, t be positive integers with $n \geq 4$, $a, b \geq 2$, $t < \min\{a, b\}$, $a + b < n + t$, and $\begin{bmatrix} n \\ a \end{bmatrix} \leq \begin{bmatrix} n \\ b \end{bmatrix}$. If $\mathcal{A} \subset \begin{bmatrix} V \\ a \end{bmatrix}$ and $\mathcal{B} \subset \begin{bmatrix} V \\ b \end{bmatrix}$ are cross-t-intersecting, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \begin{bmatrix} n \\ b \end{bmatrix} - \sum_{i=0}^{t-1} q^{(a-i)(b-i)} \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} n-a \\ b-i \end{bmatrix} + 1. \tag{4}$$

Moreover, equality holds if and only if $\mathcal{A} = \{A\}$ and $\mathcal{B} = \begin{bmatrix} V \\ b \end{bmatrix} \backslash N(A)$ where $A \in \begin{bmatrix} V \\ a \end{bmatrix}$, or $\mathcal{A} = \begin{bmatrix} V \\ b \end{bmatrix} \backslash N(B)$ and $\mathcal{B} = \{B\}$ where $B \in \begin{bmatrix} V \\ b \end{bmatrix}$, subject to $\begin{bmatrix} n \\ a \end{bmatrix} = \begin{bmatrix} n \\ b \end{bmatrix}$.

Theorem 1.4 *Let n and t be positive integers with $n \geq 4$ and $t \leq n - 2$. If*

$\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting families in S_n, then

$$|\mathcal{A}| + |\mathcal{B}| \leq n! - \sum_{i=0}^{t-1} \binom{n}{i} D_{n-i} + 1, \tag{5}$$

where D_{n-i} is the number of derangements in S_{n-i}. Moreover, equality holds if and only if $\{\mathcal{A}, \mathcal{B}\} = \{\{\sigma\}, S_n \backslash N(\sigma)\}$ where $\sigma \in S_n$.

We shall prove Theorem 1.1 in the next section, Theorem 1.2 in Section 3, Theorem 1.3 in Section 4 and Theorem 1.4 in Section 5.

2 Proof of Theorem 1.1

Before to start the proof of Theorem 1.1 we present two lemmas.

Lemma 2.1 *Let $G(X, Y)$ be a non-complete bipartite graph. Then, $|Y| - \epsilon(X) = |X| - \epsilon(Y)$, and*

(i) *$A \in \mathcal{F}(X)$ if and only if $Y \backslash N(A) \in \mathcal{F}(Y)$, and $N(Y \backslash N(A)) = X \backslash A$;*
(ii) *$A \cap B$ and $A \cup B$ are both in $\mathcal{F}(X)$ if $A, B \in \mathcal{F}(X)$, $A \cap B \neq \emptyset$ and $N(A \cup B) \neq Y$.*

Proof. Suppose $A \in \mathcal{F}(X)$ and put $C = Y \backslash N(A)$. Clearly, $N(C) \subseteq X \backslash A$. If $N(C) \neq X \backslash A$, writing $A' = X \backslash N(C)$, then $A \subsetneq A'$ and $N(A') = N(A)$. So $|N(A')| - |A'| < |N(A)| - |A| = \epsilon(X)$, yielding a contradiction. Hence $N(C) = X \backslash A$, and $|N(C)| - |C| = (|X| - |A|) - (|Y| - |N(A)|) = \epsilon(X) - |Y| + |X| \geq \epsilon(Y)$. Symmetrically, for $D \in \mathcal{F}(Y)$, putting $A = X \backslash N(D)$, we have $N(A) = Y \backslash D$ and $|N(A)| - |A| = (|Y| - |D|) - (|X| - |N(D)|) = \epsilon(Y) - |X| + |Y| \geq \epsilon(X)$. We then obtain that $\epsilon(X) + |X| = \epsilon(Y) + |Y|$ and (i) holds.

Now, suppose that $A, B \in \mathcal{F}(X)$, $A \cap B \neq \emptyset$ and $N(A \cup B) \neq Y$. Then $|N(A \cup B)| - |A \cup B| \geq \epsilon(X)$ and $|N(A \cap B)| - |A \cap B| \geq \epsilon(X)$. Note that $N(A \cup B) = N(A) \cup N(B)$ and $N(A \cap B) \subseteq N(A) \cap N(B)$. We have

$$\begin{aligned}
\epsilon(X) &\leq |N(A \cup B)| - |A \cup B| \\
&= |N(A)| + |N(B)| - |N(A) \cap N(B)| - |A| - |B| + |A \cap B| \\
&\leq 2\epsilon(X) - (|N(A \cap B)| - |A \cap B|) \leq \epsilon(X),
\end{aligned}$$

which implies that $|N(A \cup B)| - |A \cup B| = \epsilon(X)$ and $|N(A \cap B)| - |A \cap B| = \epsilon(X)$, hence (ii) holds. $\square$

From the first statement of this lemma it follows that there is a one to one correspondence $\phi : \mathcal{F}(X,Y) \mapsto \mathcal{F}(X,Y)$, where

$$\phi(A) = \begin{cases} Y\backslash N(A) \text{ if } A \in \mathcal{F}(X), \\ X\backslash N(A) \text{ if } A \in \mathcal{F}(Y). \end{cases}$$

Moreover, ϕ is an involution, i.e., $\phi^{-1} = \phi$, and $|A| + |\phi(A)| = \alpha(X,Y)$. A fragment is called *balanced* if $|A| = |\phi(A)|$. Clearly, all balanced fragments have identical size $\frac{1}{2}\alpha(X,Y)$.

Lemma 2.2 *Let $G(X,Y)$ be a non-complete and part-transitive bipartite graph under the action of a group Γ. Suppose that $A \in \mathcal{F}(X,Y)$ such that $\emptyset \neq \gamma(A) \cap A \neq A$ for some $\gamma \in \Gamma$. If $|A| \leq |\phi(A)|$, then $A \cup \gamma(A)$ and $A \cap \gamma(A)$ are both in $\mathcal{F}(X,Y)$.*

Proof. Without loss of generality, suppose $A \in \mathcal{F}(X)$ and $|A| \leq |\phi(A)| = |Y\backslash N(A)|$. Since $|N(A)| = |A| + \epsilon(X)$ and $|N(A \cap \gamma(A))| \geq |A \cap \gamma(A)| + \epsilon(X)$,

$$\begin{aligned} |N(A \cup \gamma(A))| &= 2|N(A)| - |N(A) \cap N(\gamma(A))| \\ &\leq 2|N(A)| - |N(A \cap \gamma(A))| \\ &\leq |N(A)| + |A| + \epsilon(X) - (|A \cap \gamma(A)| + \epsilon(X)) \\ &= |N(A)| + |A\backslash\gamma(A)| < |N(A)| + |Y\backslash N(A)| = |Y|. \end{aligned}$$

Then, by Lemma 2.1 (ii), $A \cap \gamma(A)$ and $A \cup \gamma(A)$ are both in $\mathcal{F}(X)$. □

From the above lemma we have that if every element of X (Y) is primitive and there is an $A \in \mathcal{F}(X)$ ($\mathcal{F}(Y)$) with $|A| \leq |\phi(A)|$, then $\mathcal{F}(X)$ ($\mathcal{F}(Y)$) contains a singleton. In particular, when $|X| = |Y|$ there are always two kinds of fragments in X: one is $\{a\}$ for $a \in X$, the other is $X\backslash N(b)$ for $b \in Y$. The former is a minimal-sized fragment, and the latter is maximal-sized one. We call the fragments of this kinds *trivial*. All others are *nontrivial*.

Proof of Theorem 1.1. From the above discussion we have that $\mathcal{F}(X)$ or $\mathcal{F}(Y)$ contains a singleton, that is, $\alpha(X,Y) = \max\{|Y| - d(X) + 1, |X| - d(Y) + 1\}$. By counting the edges of $G(X,Y)$ we have $d(X)|X| = d(Y)|Y|$, so $d(X) = d(Y)|Y|/|X| \geq d(Y)$ because $|Y| \geq |X|$. Then

$$|Y| - |X| = d(X)|X|/d(Y) - |X| = (d(X) - d(Y))|X|/d(Y) \geq d(X) - d(Y).$$

Equality holds if and only if $d(X) = d(Y)$ hence $|X| = |Y|$ because $|X| > d(Y)$. This proves that $|X| - d(Y) + 1 \leq |Y| - d(X) + 1$ and equality holds if

and only if $|X| = |Y|$. In any cases, $\alpha(X,Y) = |Y| - d(X) + 1$.

We complete the proof by two cases.

Case 1: $|X| < |Y|$. In this case we have seen that $\mathcal{F}(X)$ contains singletons while $\mathcal{F}(Y)$ does not. Now, let A be a maximal-sized element of $\mathcal{F}(X)$ and write $B = \phi(A) = Y \backslash N(A)$. Then B is a minimal-sized element of $\mathcal{F}(Y)$ with $|B| > 1$ and $\phi(B) = A$. Suppose $|A| > 1$. Since A and B are primitive, there are $\sigma, \gamma \in \Gamma$ such that $\sigma(A) \cap A \neq \emptyset$, $\sigma(A) \neq A$, $\gamma(B) \cap B \neq \emptyset$ and $\gamma(B) \neq B$. From this and Lemma 2.2 it follows that if $A| \leq |B| = |\phi(A)|$, then $\sigma(A) \cup A \in \mathcal{F}(X)$, contradicting the maximality of $|A|$; if $|B| \leq |A| = |\phi(B)|$, then $\gamma(B) \cap B \in \mathcal{F}(Y)$, contradicting the minimality of $|B|$. This proves that $|A| = 1$ for every $A \in \mathcal{F}(X)$.

Case 2: $|X| = |Y|$. In this case, if there is a nontrivial fragment in X or in Y, let A be a minimal-sized one. Then $1 < |A| \leq |\phi(A)|$. From Lemma 2.2 it follows that for every $\gamma \in \Gamma$, $\gamma(A) \cap A$ is a fragment whenever $\gamma(A) \cap A \neq \emptyset$. Then, the minimality of $|A|$ implies that $|\gamma(A) \cap A| = 0$, 1 or $|A|$, for every $\gamma \in \Gamma$, i.e., A is semi-imprimitve.

The proof is complete. $\square$

For applications of the theorem, we make further discussions on the fragments in the rest of this section.

Note that most bipartite graphs concerning here have only trivial fragments, but there are actually bipartite graphs, which have sufficiently large nontrivial fragments. For example, let n and r are fixed positive integer with $r < n$, $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Define $x_i y_j$ to be an edge of $G(X,Y)$ if and only if $j \in \{i, i+1, \ldots, i+r-1\} \pmod n$ (see Fig. 2 for $n = 5$ and $r = 3$). It is easy to verify that $\{x_i, x_{i+1}, \ldots, x_{i+j}\}$, where $1 \leq j \leq n-r-1$ and the subscripts are computed modulo n, is a fragment in X.

However, as we shall see, whether or not a bipartite graph has sufficiently large fragments depends if it has a 2-fragment.

Proposition 2.3 *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| = |Y|$ and $\epsilon(X) = d(X) - 1$, and let Γ be a group part-transitively acting on $G(X,Y)$. If there is a 2-fragment in X, then either*

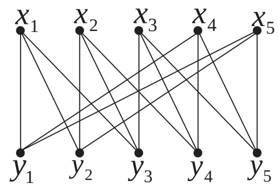

Fig. 1.

(i) *there is an imprimitive set* $A \subset X$ *with* $|N(A)| - |A| = d(X) - 1$*, or*

(ii) *there is a subset* $A \subseteq X$*, where* $A = X$ *or* A *is an imprimitive set under the action of* Γ *with* $|A| > 2$*, such that the quotient group* $\Gamma_A/(\cap_{a\in A}\Gamma_a)$ *is isomorphic to a subgroup of the dihedral group* $D_{|A|}$*, where* $\Gamma_A = \{\sigma \in \Gamma : \sigma(A) = A\}$.

Proof. By definition we have that for any $x, y \in X$, $\{x, y\}$ is a 2-fragment if and only if $|N(x) \cap N(y)| = d(X) - 1$. We now define a simple graph $H(X)$, whose vertex set $V(H)$ is X, and whose edge set $E(H)$ consists of all pairs xy's such that $\{x, y\}$ is a fragment in X. Then, each element of Γ induces an automorphism of $H(X)$. So $H(X)$ is vertex-transitive. As usual, the valency of $H(X)$ is denoted by $d(H)$.

Let H' be a connected component of $H(X)$ and let A be the vertex set of H'. Then $|A| \geq 2$. If $|A| = 2$, then A is clearly an imprimitive set in A with $|N(A)| - |A| = d(X) - 1$. Suppose $|A| > 2$ and let xyz be a path in H' for distinct $x, y, z \in A$. Set $N(x) = B \cup \{a\}$ and $N(y) = B \cup \{b\}$, where $B = N(x) \cap N(y)$. Since $yz \in E(H')$, $N(z) = (N(y) \setminus \{c\}) \cup \{d\}$ for some $c \in N(y)$ and $d \in N(z) \setminus N(y)$. If $d \in N(x) \cup N(y)$, then $N(\{x, y, z\}) = N(\{x, y\})$, contradicting that $\{x, y\}$ is a fragment. Therefore, $d \notin N(x) \cup N(y)$. So

$$N(x) \cap N(z) = \begin{cases} B, & \text{if } c = b, \\ B \setminus \{c\}, & \text{if } c \neq b. \end{cases}$$

From this it follows immediately the following.

Claim: $N(y) \subset N(x) \cup N(z)$ if the induced subgraph $H'[\{x, y, z\}]$ is a path, and $|N(x) \cap N(y) \cap N(z)| = d(X) - 1$ if $H'[\{x, y, z\}]$ is a cycle.

If H' is a complete graph, then from the above claim it follows that $|\cap_{x\in A} N(x)| = d(X) - 1$, so $|N(A)| - |A| = d(X) - 1$. Since $d(X) \geq 2$, we have $|A| < |X|$, hence A is an imprimitive set in X with $|N(A)| - |A| = d(X) - 1$.

If H' is not complete, then there are more than three elements of A, say $x_1, x_2, \ldots, x_m$, such that the induced subgraph $H'[\{x_1, \ldots, x_m\}]$ is a cycle, written as $x_1 x_2 \cdots x_m x_1$. By definition we see that $|N(\{x_1, x_2, \ldots, x_s\})| \leq d(X) - 1 + s$ for $1 \leq s \leq m-1$, and equality holds if $N(\{x_1, x_2, \ldots, x_s\}) \neq Y$, that is, $\{x_1, x_2, \ldots, x_s\}$ is a fragment. Now, if $N(\{x_1, x_2, \ldots, x_{m-1}\}) \neq Y$, then, by the above claim, $N(\{x_1, x_2, \ldots, x_m\}) = N(\{x_1, x_2, \ldots, x_{m-1}\}) \neq Y$, which yields a contradiction since $|N(\{x_1, x_2, \ldots, x_m\})| - m < d(X) - 1$. Therefore, $N(\{x_1, x_2, \ldots, x_{m-1}\}) = Y$. Assume that t is the least index such that $N(\{x_1, x_2, \ldots, x_t\}) = Y$, where $2 < t \leq m-1$. This means that every path of length less than t on this cycle is a fragment. In this case, if $d(H) > 2$, then there is an $x \in A \backslash \{x_1, \ldots, x_m\}$ such that $x x_{t-1}$ is an edge of H'. Setting $a \in N(x) \backslash N(x_{t-1})$, we have that $a \in N(x_i)$ for some $i \in [t] \backslash \{t-1\}$. Then $N(\{x_i, \ldots, x_{t-1}\}) = N(\{x_i, \ldots, x_{t-1}, x\})$ if $i < t-1$, or $N(\{x_{t-1}, x_t\}) = N(\{x_{t-1}, x_t, x\})$ if $i = t$. Both the cases contradict that $\{x_i, \ldots, x_{t-1}\}$ and $\{x_{t-1}, x_t\}$ are fragments. This proves that H' is a cycle, and hence (ii) holds. $\square$

Proposition 2.4 *Let $G(X, Y)$ be as in Proposition 2.3. If there are no 2-fragments in $\mathcal{F}(X, Y)$, then every nontrivial fragment $A \in \mathcal{F}(X)$ (if it exists) is balanced, and for each $a \in A$, there is a unique nontrivial fragment B such that $A \cap B = \{a\}$.*

Proof. Let A be a minimal-sized nontrivial fragment in X and Y. Then, $\phi(A)$ is a maximal-sized fragment in X and Y. Without loss of generality, suppose that $A \in \mathcal{F}(X)$. Then $\phi(A) = Y \backslash N(A)$ and $|Y \backslash N(A)| \geq |A|$. We now prove that the equality holds, i.e., A is balanced. Suppose, to the contrary, that $|Y \backslash N(A)| > |A|$. Set $\mathcal{A} = \{\sigma(A) : \sigma \in \Gamma\}$. As we have mentioned, A is semi-imprimitive, so $|B \cap C| = 1$ or 0 for all distinct $B, C \in \mathcal{A}$. We now define a graph $H(\mathcal{A})$, whose vertex set is $\mathcal{A}$, and whose edge set consists of all pairs BC's such that $|B \cap C| = 1$ for $A, B \in \mathcal{A}$. Clearly, $H(\mathcal{A})$ is vertex-transitive. Since A is primitive, $H(\mathcal{A})$ is not an empty graph. Suppose that $A \cap B = \{b\}$ for some $B \in \mathcal{A}$ and $b \in A$. Then, for each $a \in A$, the part-transitivity of $G(X, Y)$ implies that there is a $\sigma \in \Gamma$ with $\sigma(b) = a$, hence $\sigma(A) \cap \sigma(B) = \{a\}$. From this it follows that the valency of $H(\mathcal{A})$, denoted by $d(H)$, is at least $|A| > 2$. Hence $H(\mathcal{A})$ contains a cycle. Let $A A_1 \ldots A_s A$ be one of minimum length. Then the induced subgraph $H[\{A, A_1, \ldots, A_i\}]$ is a path from A to A_i for $i = 1, 2, \ldots, s-1$. By Lemma 2.1, if $N(A \cup A_1 \cup \cdots \cup A_i) \neq Y$, then both $A \cup A_1 \cup \cdots \cup A_i$ and $Y \backslash N(A \cup A_1 \cup \cdots \cup A_i)$ are fragments. Furthermore, if $|Y| - |N(A \cup A_1 \cup \cdots \cup A_i)| > 1$, then the minimality of A

implies $|Y| - |N(A \cup A_1 \cup \cdots \cup A_i)| \geq |A|$, hence

$$\begin{aligned}|N(A \cup A_1 \cup \cdots \cup A_{i+1})| &\leq |N(A \cup A_1 \cup \cdots \cup A_i)| + |N(A_{i+1})| \\ &\quad -|N((A \cup A_1 \cup \cdots \cup A_i) \cap A_{i+1})| \\ &= |N(A \cup A_1 \cup \cdots \cup A_i)| + |A| - 1 \leq |Y| - 1,\end{aligned}$$

i.e., $A \cup A_1 \cup \cdots \cup A_{i+1}$ is also a fragment. Now, if $|Y| - |N(A \cup A_1 \cup \cdots \cup A_{s-1})| > 1$, then, by Lemma 2.1, $A_s \cap (A \cup \cdots \cup A_{s-1})$ is a fragment. However, it is clear that $|A_s \cap (A \cup \cdots \cup A_{s-1})| = 2$, yielding a contraction. Therefore, there is a unique index k with $2 \leq k \leq s-1$ such that $|Y| - |N(A \cup A_1 \cup \cdots \cup A_k)| = 1$, that is, $A \cup A_1 \cup \cdots \cup A_k$ is a maximal-sized fragment. In this case, it is clear that $|Y| - |N(A \cup A_1 \cup \cdots \cup A_{k-1})| = |A|$. We now find a contradiction.

Set $A' = A \cup A_1 \cup \cdots \cup A_{k-1}$. Then for each $B \in (N_H(A) \cup N_H(A_{k-1})) \setminus \{A_1, A_{k-2}\}$, the induced subgraph $H[\{A, A_1, \ldots, A_{k-1}, B\}]$ is a path of length k in $H(\mathcal{A})$, so the above argument is available here. We thus obtain at least $2(d(H) - 1)$ many maximal-sized fragments in X containing A'. On the other hand, for every maximal-sized fragment $C \in \mathcal{F}(X)$ containing A', we have that $C = X \setminus N(b)$ for some $b \in Y \setminus N(A')$ since $|Y \setminus N(C)| = 1$, hence there are at most $|Y \setminus N(A')| = |A|$ many maximal-sized fragments in $\mathcal{F}(X)$ containing A', yielding a contradiction because $2(d(H) - 1) \geq 2(|A| - 1) > |A|$. This proves that $|Y \setminus N(A)| = |A|$, i.e., A is balanced. Assume $A = \{a_1, \ldots, a_d\}$ where $d > 2$. As we have seen, for each i, there is a $\sigma_i \in \Gamma$ such that $A \cap \sigma_i(A) = \{a_i\}$. Then $A \cup \sigma_i(A)$ is a maximal-sized fragment containing A, and the semi-imprimitivity and $d > 2$ imply $A \cup \sigma_i(A) \neq A \cup \sigma_j(A)$ if $i \neq j$. Therefore, $A \cup \sigma_i(A)$, $i = 1, \ldots, d$, are the all maximal-sized fragments containing A. This proves that for every $a \in A$, there is only one nontrivial fragment B with $A \cap B = \{a\}$. $\square$

3 Proof of Theorem 1.2

With the assumptions in the theorem, we put $\mathcal{X} = \binom{[n]}{a}$ and $\mathcal{Y} = \binom{[n]}{b}$. The bipartite graph $G(\mathcal{X}, \mathcal{Y})$ is defined by the cross-t-intersecting relation between $\mathcal{X}$ and $\mathcal{Y}$: For $A \in \mathcal{X}$ and $B \in \mathcal{Y}$, $AB \in E(G)$ if and only if $|A \cap B| < t$. It is easy to check that $G(\mathcal{X}, \mathcal{Y})$ is connected since $a + b < t + n$ and $(n, t) \neq (a + b, 1)$, and $G(\mathcal{X}, \mathcal{Y})$ is non-complete since $t < \min\{a, b\}$. Clearly, S_n transitively acts on $\mathcal{X}$ and $\mathcal{Y}$, respectively, in a natural way, and preserves

the cross-t-intersecting relation. Therefore, $d(\mathcal{X}) = |N(A)|$ for each $A \in \mathcal{X}$, and $d(\mathcal{Y}) = |N(B)|$ for each $B \in \mathcal{Y}$. It is easy to see that for each $A \in \mathcal{X}$,

$$N(A) = \{B \in \binom{[n]}{b} : |A \cap B| < t\} = \bigcup_{0 \le i \le t-1} \{B \in \binom{[n]}{b} : |A \cap B| = i\},$$

hence $|N(A)| = \sum_{i=0}^{t-1} \binom{a}{i}\binom{n-a}{b-i}$. Similarly, we have $|N(B)| = \sum_{i=0}^{t-1} \binom{b}{i}\binom{n-b}{a-i}$.

It is well known that for each $A \in \binom{[n]}{k}$, the stabilizer of A is a maximal subgroup of S_n subject to $n \neq 2k$ [3]. Therefore, the action of S_n on $\binom{[n]}{k}$ is imprimitive if and only if $n = 2k \geq 4$, and the only imprimitive sets are all pairs of complementary subsets. If $n = 2a \geq 4$ and $(n, t) \neq (a + b, 1)$, then from $a + b < n + t$ it follows $b < a + t$. For every pair A and $\overline{A}$ in $\binom{[n]}{a}$, it is easy to verify that $\{C \cup \{i\} : C \in \binom{\overline{A}}{a-1}, i \in A\} \subseteq N(A) \backslash N(\overline{A})$ if $b = a$ and $t > 1$, $\binom{\overline{A}}{b} \subseteq N(A) \backslash N(\overline{A})$ if $b < a$, and $\{\overline{A} \cup C : C \in \binom{A}{b-a}\} \subseteq N(A) \backslash N(\overline{A})$ if $b > a$. This implies $|N(A) \backslash N(\overline{A})| > 1$ hence $|N(A) \cap N(\overline{A})| < |N(A)| - 1$. Therefore, $\{A, \overline{A}\}$ is not a fragment for every $A \in \binom{[n]}{a}$. We thus prove that every fragment in $\mathcal{X}$ and $\mathcal{Y}$ is primitive. Then, by Theorem 1.1, inequality (3) holds.

To complete the proof of Theorem 1.2 we need to determine all nontrivial fragments. Suppose there is a nontrivial fragment in $\binom{[n]}{a}$ or $\binom{[n]}{b}$. Without loss of generality we assume that $\mathcal{S}$ is a minimal-sized one in $\binom{[n]}{a}$. By Theorem 1.1, $\binom{n}{a} = \binom{n}{b}$, i.e., $b = a$ or $b = n - a$. Clearly, S_n is not isomorphic to a subgroup of $D_{n!}$ for $n \geq 4$. Therefore, by Proposition 2.3, there are no 2-fragment in $\mathcal{F}(\mathcal{X})$ and $\mathcal{F}(\mathcal{Y})$, which implies that $\mathcal{S}$ is balanced.

For each $C \subseteq [n]$, S_C is embedded into S_n in a natural way: for $\sigma \in S_C$, let σ fixes elements of $\overline{C}$. Now, take a $C \in \mathcal{S}$ and let $\Gamma = S_C \times S_{\overline{C}}$ and $\Gamma_{\mathcal{S}} = \{\sigma \in \Gamma : \sigma(\mathcal{S}) = \mathcal{S}\}$. Then $C \in \sigma(\mathcal{S})$ for each $\sigma \in \Gamma$. Since $\mathcal{S}$ has more than one elements, we have $\Gamma_{\mathcal{S}} \neq \Gamma$. Otherwise, $S_C \times S_{\overline{C}}$ and $S_B \times S_{\overline{B}}$ (for some $B \in \mathcal{S} \backslash \{C\}$) will generate whole S_n so that $S = \binom{[n]}{a}$, yielding a contradiction. Then, by Proposition 2.4 we have that $[\Gamma : \Gamma_{\mathcal{S}}]$, the index of $\Gamma_{\mathcal{S}}$ in Γ, equals 2. Now, let $\Gamma_{\mathcal{S}}[C]$ be the projection of $\Gamma_{\mathcal{S}}$ onto S_C. Then, $\Gamma_{\mathcal{S}}[C]$ is a subgroup of S_C of index ≤ 2. That is, $\Gamma_{\mathcal{S}}[C] = S_C$ or A_C. From this we see that $A_C \times S_{\overline{C}}$ and $S_C \times A_{\overline{C}}$ are the only index-2 subgroups of Γ. That is, $\Gamma_{\mathcal{S}} = A_C \times S_{\overline{C}}$ or $S_C \times A_{\overline{C}}$. For any $B \in \mathcal{S} \backslash \{C\}$, $a = |B \cap C| + |B \cap \overline{C}|$. If $|B \cap C| > 1$, let (i, j) be an interchange, where $i, j \in B \cap C$. Then, (i, j) fixes both C and B. The semi-imprimitivity of $\mathcal{S}$ implies $(i, j) \in \Gamma_{\mathcal{S}}$. This yields

$\Gamma_{\mathcal{S}} = S_C \times A_{\overline{C}}$. From this process it follows that, for each $B \in \mathcal{S}$, there exists at most one of $|B \cap C|$ and $|B \cap \overline{C}|$ being grater than 1. Note that if $B \subseteq \overline{C}$, then S_C and S_B fix both C and B, i.e., $S_C \times S_B \subseteq \Gamma_{\mathcal{S}}$. It is clear, however, that neither $A_C \times S_{\overline{C}}$ nor $S_C \times A_{\overline{C}}$ contain $S_C \times S_B$. We therefore obtain that $|C \cap B| = 1$ for every $B \in \mathcal{S}$, or $|C \cap B| = a - 1$ for every $B \in \mathcal{S}$.

Suppose $|C \cap B| = 1$ for every $B \in \mathcal{S}$. Without loss of generality we assume $C \cap B = \{1\}$ for some $B \in \mathcal{S}$. In this case, if $a > 2$, then $|B \cap \overline{C}| \geq 2$, so $\Gamma_{\mathcal{S}} = A_C \times S_{\overline{C}}$. On the other hand, we can find distinct $i, j \in C$ such that $(1, i, j)(B) = B \backslash \{1\} \cup \{i\} \in \mathcal{S}$ because $(1, i, j) \in A_C$. From this it follows that $(1, i)(\mathcal{S})$ contains more than one element of $\mathcal{S}$, hence $(1, i) \in \Gamma_{\mathcal{S}}$. The contradiction proves $a = 2$. Thus $\mathcal{S}$ consists of all 2-subsets $\{1, i\}$'s for $i \in [2, n]$. Since $t < \min\{a, b\}$ and $(n, t) \neq (a + b, 1)$, we have $t = 1$ and $b = 2$. Then $d(\mathcal{X}) = \binom{n-2}{2}$ and $N(\mathcal{S}) = \binom{[2,n]}{2}$ satisfying $|N(\mathcal{S})| - |\mathcal{S}| = d(\mathcal{X}) - 1$, that is, $\mathcal{S}$ is a fragment in $\binom{[n]}{2}$.

Suppose now $|C \cap B| = a - 1 > 1$ for every $B \in \mathcal{S}$. In this case, we may similarly prove that $n - a = 2$, $b = a$, $t = n - 3$ and $\Gamma_{\mathcal{S}} = S_C$. Thus $\mathcal{S} = \{\sigma(B \cap C) \cup \{i\} : \sigma \in S_C \text{ and } \{i\} = B \cap \overline{C}\} \cup \{C\} = \binom{A}{n-2}$ where $A = B \cup C$ is a $(a + 1)$-subset of $[n]$. It is easy to verify that $\mathcal{S}$ is a fragment in $\binom{[n]}{n-2}$. $\square$

4 Proof of Theorem 1.3

Similarly to the proof of Theorem 1.2, put $\mathcal{X} = \begin{bmatrix} V \\ a \end{bmatrix}$ and $\mathcal{Y} = \begin{bmatrix} V \\ b \end{bmatrix}$. The bipartite graph $G(\mathcal{X}, \mathcal{Y})$ is defined by the cross-t-intersecting relation between $\mathcal{X}$ and $\mathcal{Y}$: for $A \in \mathcal{X}$ and $B \in \mathcal{Y}$, $(A, B) \in E(G)$ if and only if $\dim(A \cap B) < t$. Analogously to the families of sets, $G(\mathcal{X}, \mathcal{Y})$ is connected and non-complete. Let $GL(V)$ denote the general linear group of V, which consists of all invertible linear transformations of V. Clearly, $GL(V)$ transitively acts on $\mathcal{X}$ and $\mathcal{Y}$, respectively, in a natural way, and preserves the cross-t-intersecting relation. So $d(\mathcal{X}) = |N(A)|$ for $A \in \mathcal{X}$. It is easy to see that

$$N(A) = \bigcup_{i=0}^{t-1} N_i^b(A),$$

where $N_i^b(A) = \left\{T \in \begin{bmatrix} V \\ b \end{bmatrix} : \dim(T \cap A) = i\right\}$. To determine $|N(A)|$, we need a usefull result, stated as a lemma as follows.

Lemma 4.1 *([9, Proposition 2.2]) Let $A \in \begin{bmatrix} V \\ a \end{bmatrix}$. Then $|N_0^b(A)| = \begin{bmatrix} n-a \\ b \end{bmatrix} q^{ab}$.*

Suppose $i > 0$ and let C be an arbitrary i-subspace of A. We consider the quotient space V/C. Then $\dim(V/C) = n - i$, $\dim(A/C) = a - i$ and $|N_0^{b-i}(A/C)| = \begin{bmatrix} n-a \\ b-i \end{bmatrix} q^{(a-i)(b-i)}$. So $|N_i^b(A)| = \begin{bmatrix} a \\ i \end{bmatrix} |N_0^{b-i}(A/C)| = \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} n-a \\ b-i \end{bmatrix} q^{(a-i)(b-i)}$ (See also [28, Lemma 4] and [25, Lemma 2.4]). Thus

$$|N(A)| = \sum_{i=0}^{t-1} q^{(a-i)(b-i)} \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} n-a \\ b-i \end{bmatrix} = d(\mathcal{X}).$$

Similarly, $d(\mathcal{Y}) = \sum_{i=0}^{t-1} q^{(a-i)(b-i)} \begin{bmatrix} b \\ i \end{bmatrix} \begin{bmatrix} n-b \\ a-i \end{bmatrix}$.

For a subspace A of V, by $GL(V|A)$ we denote the stabilizer of A in $GL(V)$. It is well known that for $A \in \mathcal{X}$, $GL(V|A)$ is a maximal subgroup of $GL(V)$ [1], so the action of $GL(V)$ on $\mathcal{X}$ is primitive. Then, by Theorem 1.1, inequality (4) holds, and each nontrivial fragment is a semi-imprimitive set under the action of $GL(V)$.

To complete the proof of Theorem 1.2 we need to determine all nontrivial fragments. Suppose there is a nontrivial fragment in $\mathcal{X}$ or $\mathcal{Y}$. Without loss of generality we assume that $\mathcal{S}$ is a minimal-sized one in $\mathcal{X}$. By Theorem 1.1, $\begin{bmatrix} n \\ a \end{bmatrix} = \begin{bmatrix} n \\ b \end{bmatrix}$, i.e., $b = a$ or $b = n - a$. Clearly, $GL(V)/K$ is not isomorphic to a subgroup of $D_{|\mathcal{X}|}$, where K is the kernel of the action of $GL(V)$ on $\begin{bmatrix} V \\ a \end{bmatrix}$ or $\begin{bmatrix} V \\ b \end{bmatrix}$. Therefore, by Proposition 2.3, there are no 2-fragment in $\mathcal{F}(\mathcal{X}, \mathcal{Y})$, which implies that $\mathcal{S}$ is balanced.

Take $C \in \mathcal{S}$, write $\Gamma = GL(V|C)$ and $\Gamma_{\mathcal{S}} = \{\sigma \in \Gamma : \sigma(\mathcal{S}) = \mathcal{S}\}$. Then, $\Gamma \neq \Gamma_{\mathcal{S}}$, and again by Proposition 2.3, $[\Gamma, \Gamma_{\mathcal{S}}] = 2$ so that $\Gamma = \Gamma_{\mathcal{S}} \cup \gamma\Gamma_{\mathcal{S}}$ for some $\gamma \in \Gamma$. Thus $\mathcal{S}$ and $\gamma(\mathcal{S})$ are the only nontrivial fragments containing C. From the structure of Γ it follows that Γ is transitive on $N_i^a(C)$ for each $i = 0, 1, \ldots, a$, whenever $N_i^a(C) \neq \emptyset$. Set $\mathcal{S}_i = \mathcal{S} \cap N_i^a(C)$ for $0 \leq i < a$. If $\mathcal{S}_i \neq \emptyset$, then

$$N_i^a(C) = \mathcal{S}_i \cup \gamma(\mathcal{S}_i) = \left\{L + R : \; L \in \begin{bmatrix} C \\ i \end{bmatrix} \text{ and } R \in N_0^{a-i}(C)\right\}.$$

From this we see that $\Gamma_{\mathcal{S}}$ is transitive on $\mathcal{S}_i$. It is clear that the restriction of Γ on C is $GL(C)$. Therefore, the induced action of Γ on $\begin{bmatrix} C \\ i \end{bmatrix}$ is primitive, thus the action of $\Gamma_{\mathcal{S}}$ on $\begin{bmatrix} C \\ i \end{bmatrix}$ is transitive. This means that if $L_0 + R_0 \in \mathcal{S}_i$ for some $R_0 \in N_0^{a-i}(C)$, then $L + R_0 \in \mathcal{S}_i$ for every $L \in \begin{bmatrix} C \\ i \end{bmatrix}$. We complete the proof by two cases.

Case 1: $n-a=a-i$. Suppose that $L_0+R_0 \in \mathcal{S}_i$ and $\{\alpha_1, \dots, \alpha_{a-i}\}$ is a basis of R_0. Then bases of elements of $N_0^{a-i}(C)$ are of the form $\{\alpha_1+\beta_1, \dots, \alpha_{a-i}+\beta_{a-i}\}$, where $\beta_i \in C$. Put $Q = \{R \in N_0^{a-i}(C) : L+R \in \mathcal{S}_i \text{ for some } L \in \begin{bmatrix} C \\ i \end{bmatrix}\}$. Then $R_0 \in Q$. For given $\beta_1, \dots, \beta_{a-i} \in C$, let R_j be the subspace generated by $\alpha_1+\beta_1, \dots, \alpha_j+\beta_j, \alpha_{j+1}, \dots, \alpha_{a-i}$. Assume $R_j \in Q$. Then the above discussion implies $L+R_j \in Q$ for every $L \in \begin{bmatrix} C \\ i \end{bmatrix}$. Thus, we can take an $L \in \begin{bmatrix} C \\ i \end{bmatrix}$ containing β_{j+1} so that $L+R_j = L+R_{j+1}$, that is, $R_{j+1} \in Q$. This proves $\mathcal{S}_i = N_i^a(C)$, yielding a contradiction.

Case 2: $n-a > a-i$. Consider the natural map ν from V onto the quotient space V/C, that is, $\nu(A) = (A+C)/C$, written as $\bar{A}$, for any subspace A of V. Then $\nu(N_i^{a-i}(C)) = \begin{bmatrix} V/C \\ a-i \end{bmatrix}$. It is clear that Γ acts on V/C and Γ/K is isomorphic to $GL(V/C)$, where K is the kernel of the action. Then the primitivity of the action implies that $\Gamma_{\mathcal{S}}K/K$ is transitive on $\begin{bmatrix} V/C \\ a-i \end{bmatrix}$. This means that for each $\bar{R}_0 \in \begin{bmatrix} V/C \\ a-i \end{bmatrix}$, there is an $R_0 \in N_0^{a-i}(C)$ such that $\nu(R_0) = \bar{R}_0$ and $L_0+R_0 \in \mathcal{S}_i$ for some $L_0 \in \begin{bmatrix} C \\ i \end{bmatrix}$. Then, by Case 1 we prove $\mathcal{S}_i = N_i^a(C)$, yielding a contradiction, again.

We thus prove that the graph has no nontrivial fragments. $\square$

5 Proof of Theorem 1.4

We first prove a general result. Let Γ be a transitive permutation group on Ω with the identity 1. By the group and a positive integer t with $1 \le t \le |\Omega|-2$ we define a simple graph, written as $G_t = G_t(\Gamma)$, whose vertex set is Γ, and whose edge set consists of all pairs $\sigma\tau$ such that $|\{x \in \Omega : \sigma(x) = \tau(x)\}| < t$. Let Γ_L and Γ_R denote the left and right regular action on Γ, respectively. Then $\Gamma_L \times \Gamma_R$ (not necessarily a direct product) induces an automorphism group of $G_t(\Gamma)$. In a natural way, we can view $G_t(\Gamma)$ as a bipartite graph $G_t(\Gamma, \Gamma)$, which is part-transitive under the action of Γ_L and Γ_R.

Lemma 5.1 *Suppose that A is an imprimitive set in Γ under the action of $\Gamma_L \times \Gamma_R$. Then A is a coset of a non-trivial normal subgroup of Γ.*

Proof. Since A is an imprimitive set, we have that $1 < |A| < |\Gamma|$, and for every $\alpha \in \Gamma$, αA is also an imprimitive set. Without loss of generality we assume that $1 \in A$. From this it follows that $\alpha \in \alpha A$ and $1 \in \alpha^{-1}A$ for each $\alpha \in A$, hence $\alpha A = \alpha^{-1}A = A$, which implies that A is a subgroup Γ. Furthermore,

for every $\gamma \in \Gamma$, $1 \in (\gamma^{-1}A\gamma) \cap A$, hence $\gamma^{-1}A\gamma = A$, proving that A is a normal subgroup of Γ. $\square$

We now consider the graph $G_t(S_n)$ where $n \geq 4$ and $1 \leq t \leq n-2$. For $0 \leq i < n$, by $\mathcal{D}_n^i$ we denote the set of all permutations in S_n which have exact i fixed points. The elements of $\mathcal{D}_n^0$ are known for the derangements of $[n]$. As usual, set $|\mathcal{D}_n^0| = D_n$. By definition, $G_t(S_n)$ is the Cayley graph on S_n generated by $\mathcal{G}_t$, where $\mathcal{G}_t = \cup_{i=0}^{t-1}\mathcal{D}_n^i$. (cf. [21]). It is not difficult to compute that for every $\sigma \in S_n$,

$$|N(\sigma)| = |\mathcal{G}_t\sigma| = |\mathcal{G}_t| = \sum_{i=0}^{t-1}\binom{n}{i}D_{n-i}.$$

Let $\mathcal{S}$ be a fragment in S_n. Then for any $\sigma \in S_n$, $\sigma\mathcal{S}$ is also a fragment. Without loss of generality, we assume that $1 \in \mathcal{S}$ and set $\mathcal{S}^* = \mathcal{S}\backslash\{1\}$. By definition we have that $|N(\mathcal{S})| = |\mathcal{G}_t\mathcal{S}| = |\mathcal{G}_t| + |\mathcal{S}^*|$, that is

$$|\mathcal{G}_t\mathcal{S}^*\backslash\mathcal{G}_t| = |\mathcal{S}^*|. \tag{6}$$

If $\mathcal{S}$ is imprimitive, then Lemma 5.1 implies that $\mathcal{S}$ is a nontrivial normal subgroup of S_n. It is well known that the only nontrivial normal subgroups of S_n are A_n and the quaternary group $V_4 = \{1, (12)(23), (13)(24), (14)(23)\}$ for $n = 4$. Since A_n has index 2 in S_n and $\mathcal{G}_t \not\subset A_n$, $\mathcal{G}_tA_n = S_n$, hence A_n is not a fragment in S_n for $n \geq 4$. And, for $n = 4$ and $t = 1, 2$, it is straightforward to verify that $|\mathcal{G}_tV_4^* \setminus \mathcal{G}_t| > 3$, so V_4 is not a fragment in S_4. We thus prove that every fragment in S_n is primitive. Then, by Theorem 1.1 we obtain inequality (5). Moreover, by Proposition 2.3, it is easy to verify that $G_t(\Gamma, \Gamma)$ has no 2-fragments.

Suppose that there is a nontrivial fragment $\mathcal{S}$ in S_n. Then, by Proposition 2.4, $\mathcal{S}$ is balanced and $|\mathcal{S}| > 2$. Without loss of generality we may assume $1 \in \mathcal{S}$. Set $H = \{h \in S_n : h\mathcal{S} = \mathcal{S}\}$. Clearly, H is a subgroup of S_n. If $H = \{1\}$, then $\sigma\mathcal{S} = \tau\mathcal{S}$ implies $\sigma = \tau$ for any $\sigma, \tau \in S_n$, hence for any distinct $a, b \in \mathcal{S}$, by the semi-imprimitivity of $\mathcal{S}$, we have $a^{-1}\mathcal{S} \cap b^{-1}\mathcal{S} = \{1\}$. We thus obtain more than 2 $|\mathcal{S}|$-fragments containing 1, contradicting Proposition 2.4. Therefore, $|H| > 1$ and $S = \cup_{b\in S}Hb$. For each $a \in \mathcal{S}$, it is evident that $Ha \subset \mathcal{S} \cap \mathcal{S}a$. So the semi-imprimitivity of $\mathcal{S}$ implies that $\mathcal{S} = \mathcal{S}a$, which implies that $\mathcal{S}$ is a subgroup of S_n. We have seen that $\mathcal{S}$ is not normal. i.e., there is a $\sigma \in S_n$ with $\sigma^{-1}\mathcal{S}\sigma \neq \mathcal{S}$. However, each $\sigma^{-1}\mathcal{S}\sigma$ contains 1. Again

by Proposition 2.4, the normalizer $N_{S_n}(\mathcal{S})$ is an index-2 subgroup of S_n, i.e., $N_{S_n}(\mathcal{S}) = A_n$ because A_n is the only index-2 subgroup of S_n. So $\mathcal{S}$ is a normal subgroup of A_n. It is well known that A_n is a simple group for $n \geq 5$, therefore A_n has no nontrivial normal subgroup for $n \geq 5$, and A_4 has the only nontrivial normal subgroup V_4. We have seen that neither A_n nor V_4 are fragments of $G_t(\Gamma, \Gamma)$. We thus prove that the graph has no nontrivial fragments. This completes the proof. $\square$

References

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, Invent. Math. 76 (1984) 469–514.

[2] D. Ellis, E. Friedgut, H. Pilpel, Intersecting families of permutations, available online at: http://www.ma.huji.ac.il/∼ehudf/docs/tintersectingfinal.pdf.

[3] B. Newton, B. Benesh, A classification of certain maximal subgroups of symmetric groups, J. Algebra 304 (2006) 1108–1113.

[4] P. Borg, Intersecting and cross-intersecting families of labeled sets, Electron. J. Combin. 15 (2008) N9.

[5] P. Borg, A short proof of a cross-intersection theorem of Hilton, Discrete Math. 309 (2009) 4750–4753.

[6] P. Borg, Cross-intersecting families of permutations, J. Combin. Theory Ser. A, 117 (2010) 483–487.

[7] P. Borg and I. Leader, Multiple cross-intersecting families of signed sets, J. Combin. Theory Ser. A, 117 (2010) 583–588.

[8] P.J. Cameron, C.Y. Ku, Intersecting families of permutations, European J. Combin. 24 (2003) 881–890.

[9] W.Y.C. Chen, G.C. Rota, q-Analogs of the inclusionCexclusion principle and permutations with restricted position, Discrete Math. 104 (1992) 7–22.

[10] M. Deza, P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Combin. Theory Ser. A 22 (1977) 352–360.

[11] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. 2 (1961) 313–318.

[12] P. Frankl, The Erdős-Ko-Rado theorem is true for $n = ckt$, Col. Soc. Math. J. Bolyai 18 (1978) 365–75.

[13] P. Frankl and N. Tohushige, Some best possible inequalities concerning cross-intersecting families, J. Combin. Theory Ser. A 61 (1992) 87–97.

[14] P. Frankl, R.M. Wilson, The Erdős-Ko-Rado theorem for vector spaces, J. Combin. Theory Ser. A 43 (1986) 228–236.

[15] C. Greene, D. J. Kleitman, Proof techniques in the ordered sets, in: G.-C. Rota, ed., "Studies in Combinatorics" (Math. Assn. America, Washington DC, 1978) 22–79.

[16] C. Godsil, K. Meagher, A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations, European J. Combin. 30 (2009) 404–414.

[17] A.J.W. Hilton, An intersection theorem for a collection of families of subsets of a finite set, J. London Math. Soc. 2 (1977) 369–376.

[18] A.J.W. Hilton and E.C. Milner, Some intersection theorems for systems of finite sets, Quart. J. Math. Oxford 18 (1967) 369–384.

[19] W. N. Hsieh, Intersection theorems for systems of finite vector spaces, Discrete Math. 12 (1975) 1–16.

[20] N. Jacobson, Basic algebra. I, Second edition. W. H. Freeman and Company, New York, 1985.

[21] C.Y. Ku, D.B. Wales, Eigenvalues of the derangement graph, J. Combin. Theory Ser. A 117 (2010) 289–312.

[22] C.Y. Ku, I. Leader, An Erdös-Ko-Rado theorem for partial permutations, Discrete Math. 306 (2006) 74–86.

[23] B. Larose, C. Malvenuto, Stable sets of maximal size in Kneser-type graphs, European J. Combin. 25 (2004) 657–673.

[24] Y.S. Li, J. Wang, Erdös-Ko-Rado-Type Theorems for Colored Sets, Elecron. J. Combin. 14 (2007) R1.

[25] J. Wang, H.J. Zhang, Normalized matching property of a class of subspace lattices, Taiwanese J. Math. 11 (2007) 43–50.

[26] J. Wang, H.J. Zhang, Cross-intersecting families and Primitivity of symmetric systems, J. Combin. Theory Ser. A 118 (2011) 455–462.

[27] J. Wang, S.J. Zhang, An Erdős-Ko-Rado-type theorem in Coxeter groups, European J. Combin. 29 (2008) 1112–1115.

[28] Y. Wang, On a class of subspace lattices, J. Math. Res. Exposition 19 (1999) 341–348.

[29] R.M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, Combinatorica 4 (1984) 247–57.